Nonadditive entropies

Generalising Boltzmann's approach to thermodynamics

In the early 20th century, special and general relativity and quantum mechanics emerged from the revelation that classical physics can't explain everything about the nature of our universe. Yet, despite the widespread acceptance of these ideas, a similar shakeup of classical physics has never been seen in thermodynamics, as connected to the microscopic world by Boltzmann. In a new study, Hans Haubold at the Vienna National Centre, Austria, together with Constantino Tsallis at the Brazilian Center for Research in Physics, explore some of the implications of a generalised approach to thermodynamics, first developed by Tsallis in the 1980s.

Today, the concepts studied by theoretical physicists encompass a variety of fields – but ultimately, they can be boiled down to just a handful of fundamental phenomena. Hans Haubold at the Vienna National Centre illustrates: 'The pillars of contemporary theoretical physics are classical mechanics, Maxwell electromagnetism, relativity, quantum mechanics, and statistical thermodynamics'.

The first of these pillars to emerge was Newton's classical mechanics. By drawing together centuries of observations made by astronomers and natural philosophers as they examined the world around them, Newton derived robust laws to describe the mechanical motions of massive, moving bodies. Later on, in the 19th century, James Clerk Maxwell drew up a new formidable set of equations to describe the characteristics of light, and the interactions taking place between charged particles.

Through his work, Einstein proved that while Newton's ideas hold true when objects aren't travelling too fast, more generalised descriptions are required as they approach the speed of light. In his new theory of general relativity, Einstein unified Newton's mechanics with the equations first derived by Maxwell just a few decades earlier.

In parallel with Einstein, other physicists in the early 20th century were quickly discovering that Newton's mechanics also didn't hold up on very small scales: a concept which culminated in the emergence of quantum mechanics.

Figure 1. Ludwig Eduard Boltzmann (left, 1844-1906) and Josiah Willard Gibbs (right, 1839–1903).

Ultimately, these groundbreaking ideas painted a picture of interconnectedness between the pillars described by Haubold: where classical physics emerges only under the right conditions. Yet, as his collaborator Constantino Tsallis at the Brazilian Center for Research in Physics first pointed out, thermodynamics stands out as a classical theory which hasn't been generalised in the same way as Newton or Maxwell's ideas.

MOVING BEYOND CLASSICAL THERMODYNAMICS

In statistical thermodynamics, physical systems are described as occupying specific 'microstates' – featuring specific numbers of particles, each with their own specific energies and contained within specific volumes. As first described by Ludwig Boltzmann, the number of possible microstates associated with a system exhibits a simple logarithmic relationship with its 'entropy': a quantity commonly used as a measure of disorder or randomness.

Just like Newton and Maxwell's ideas, statistical thermodynamics holds up remarkably well in classical scenarios, such as the slow flow of air inside a room or the transfer of heat between two solid objects. Recently, however, the descriptive capabilities of thermodynamics in its current form have been called into question.

'In recent decades, statistical thermodynamics has started to exhibit failures or inadequacies in an increasing number of complex systems', Haubold explains. In the 1980s, Tsallis was one of the first to argue that our current conception of thermodynamics must undergo the same treatment as Newton and Maxwell's physics in the early 20th century.

INTRODUCING: TSALLIS ENTROPY

The core of Tsallis' idea is that, in complex systems, the occurrence of one microstate will strongly depend on the occurrence of another. In this case, both states are said to be 'strongly correlated'.

As a simple example, we can take four words – 'one', 'many', 'cat', and 'cats' – and ask how many pairs of words we can make with them. In theory, a total of 16 pairs can be formed. However, under the rules imposed by English grammar, strong correlations exist between pairs of words in specific orders, while others aren't correlated at all. This ultimately means that most pairs – including 'one cats', 'cat many', and 'many one' – don't make any sense. In total, just two pairs of words are possible: 'one cat' and 'many cats'.

In a more physical example, we could think of the microstates occupied by water molecules inside a whirlpool. According to classical laws of entropy, the paths taken by the molecules can, in theory, fluctuate randomly in any direction. Overall, however, the molecules must each spiral towards the centre of the whirlpool, introducing correlations between their motions.

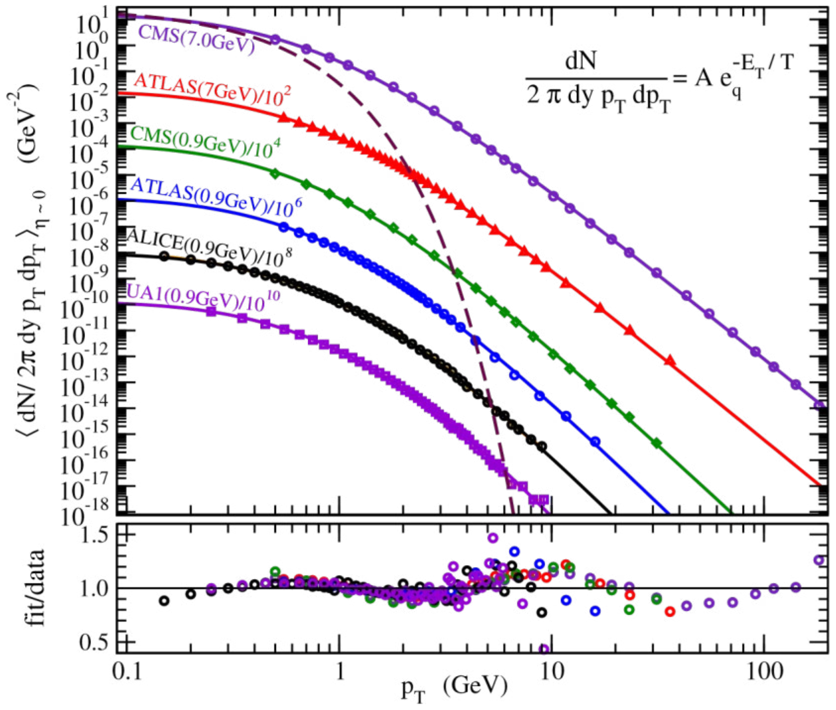

Photo Source: Wong CY, et al. (2015), doi.org/10.1103/PhysRevD.91.114027

Figure 2. Comparison of the experimental transverse momentum distribution of hadrons in *pp* collisions at central rapidity *y* with theoretical *q*-exponentials with $q = 1.13 \pm 0.02$ and $T = (0.14 \pm 0.01)$ *GeV*. The corresponding Boltzmann-Gibbs (purely exponential) fit is illustrated as the dashed curve. For a better visualisation, both the data and the analytical curves have been divided by a constant factor as indicated. The ratios data/fit are shown at the bottom, where a nearly log-periodic behaviour is observed on top of the *q*-exponential one.

In recent decades, physicists have discovered many more complex scenarios where these same principles apply. 'This wide range of important systems eventually gave support, since 1988, to the generalisation of statistical thermodynamics', Haubold describes. 'Since their introduction, these 'nonadditive' entropies and their consequences have been intensively studied worldwide.'

To represent these nonadditive entropies, Tsallis incorporated a new term into Boltzmann's original entropy formula, which quantifies the degree of correlation between a system's microstates. This new theory hasn't come without opposition within the wider physics community, but for Tsallis and his proponents, its implications for thermodynamics could be no less relevant than the ideas first introduced by Einstein and the early pioneers of quantum mechanics.

BLACK HOLES AND MEDICAL IMAGES

Using Tsallis' updated formulas, researchers have now explored its potential applications across a diverse array of scenarios. 'The emergence of such intriguing features became

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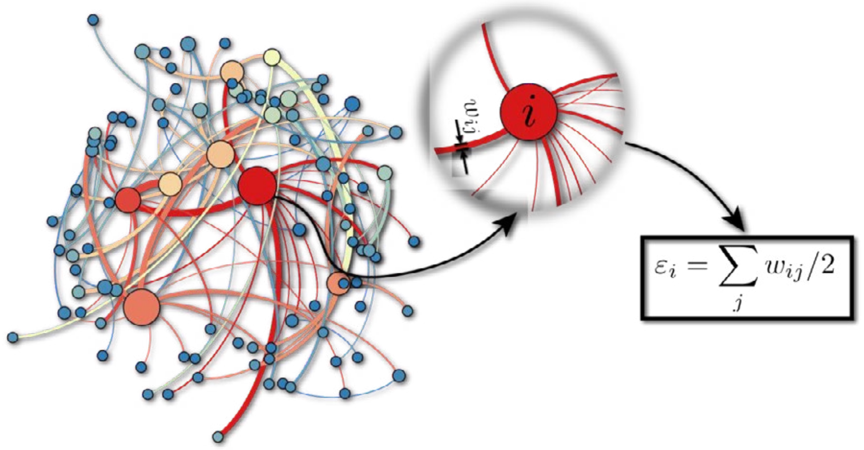

Photo Source: de Oliveira, RM, et al. (2021), doi.org/10.1038/s41598-020-80939-1

Figure 3. Sample of a $N = 100$ network for $(d, \alpha_A, \alpha_G, \eta, w_0) = (2,1,5,1,1)$. As can be seen, for this choice of parameters, hubs (highly connected nodes) naturally emerge in the network. Each link has a specific width w_{ij} and the total energy ε_i associated to the site *i* will be given by half of the sum over all link widths connected to the site *i* (see zoom of site *i*).

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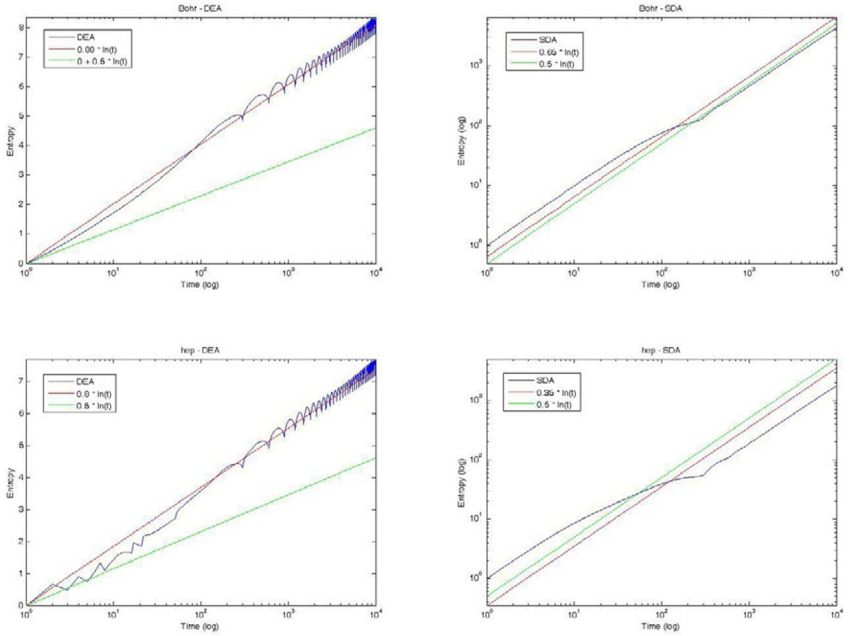

Photo Source: Mathai, AM, Haubold, HJ, (2018), doi.org/10.1140/epjb/e2017-80371-5

Figure 4. Diffusion entropy analysis with Boltzmann entropy measure.

apparent in quantum systems, such as black holes', Haubold illustrates.

Beyond a nearly spherical shell, named the 'event horizon', the laws of relativity show how black holes will bend the fabric of spacetime to such a degree that not even massless light particles can escape. According to the 'area theory' first proposed by Stephen Hawking, a black hole's event horizon can never shrink over time: a feature which Haubold and Tsallis hope that nonadditive entropy could one day help to explain.

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'In a different arena, the efficiency of the "Shannon entropy" started to be perceived as not necessarily optimal in the processing of medical images', Haubold continues. Derived by celebrated computer scientist Claude Shannon, this form of entropy uses Boltzmann's concepts to quantify the uncertainty associated with bits of information: in this case, the pixels which make up computer images.

By adapting Shannon's ideas to account for correlations between the pixels in medical images (breast cancer, multiple sclerosis, COVID-19), researchers better account for the immense complexity of the human body. In turn, Tsallis' ideas could help doctors to diagnose and treat their patients more effectively.

ILLUSTRATING NEW APPLICATIONS

In their latest paper, Haubold and Tsallis have extended this exploration even further. 'The present review focuses on these concepts and their predictions, verifications, and applications in physics and elsewhere', Haubold summarises.

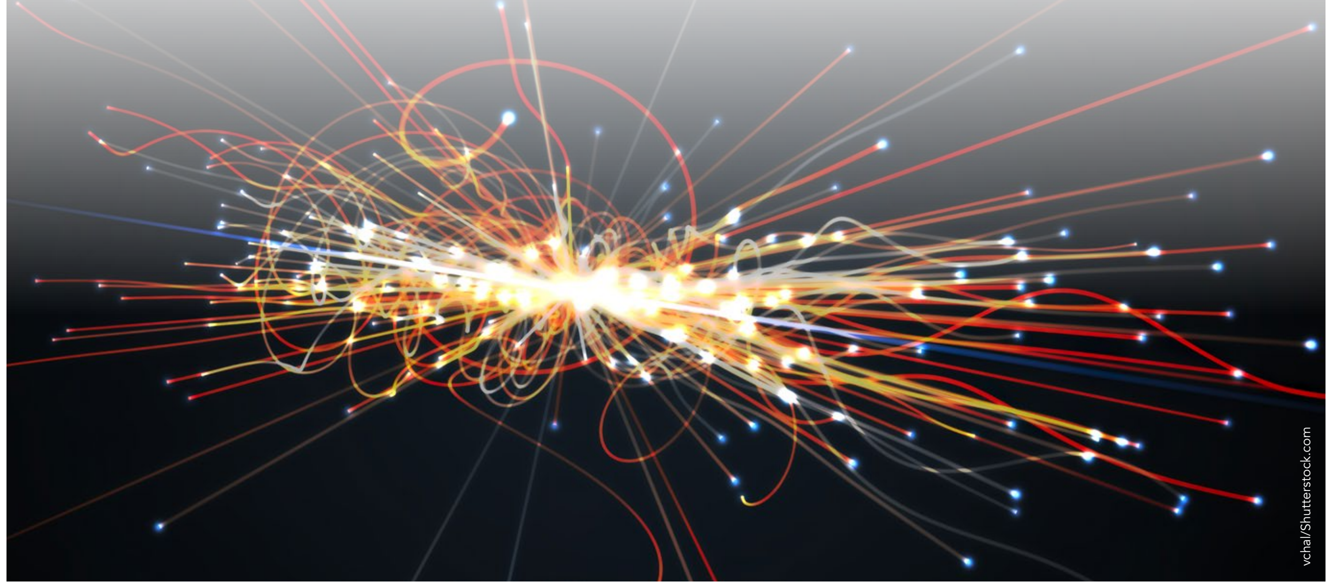
In one example, Haubold used Tsallis' formula to study the entropy associated with solar neutrinos: chargeless, almost massless particles which originate from the Sun. These particles can freely pass through the empty spaces between atomic nuclei and their orbiting electrons without ever colliding with other particles – making them notoriously difficult to detect. By quantifying correlations between their microstates, physicists could shed new light on their enigmatic origins and characteristics.

Elsewhere, amazingly precise experiments have focused on the momentum distributions of particles which emerge as beams of protons are smashed together at close to the speed of light, within particle accelerators such as CERN's Large Hadron Collider (LHC). Through more advanced methods reflecting the relevance of the nonadditive entropy of the systems which emerge following these collisions, physicists could gain a better understanding of the fundamental building blocks of our universe.

BRINGING BOLTZMANN IN LINE WITH NEWTON

Just like the laws put forward by Newton and Maxwell, Boltzmann's statistical thermodynamics is almost perfectly well suited to describing the entropy of many classical systems. Yet as our understanding of physics continues to improve, Haubold and Tsallis believe that an increasing number of scenarios have emerged for which a more generalised theory of thermodynamics is crucial for understanding how they behave.

By accounting for the spatiotemporal correlations clearly seen between the microstates of many more complex systems, the duo hopes that Tsallis' updated formula will become more accepted by the global physics community in the future. This could provide statistical thermodynamics with the same treatment which was once applied to classical mechanics and electromagnetism, which ultimately proved to transform our understanding of how nature works.



Behind the Research

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Research Objectives

Haubold and Tsallis explore a generalised approach to thermodynamics.

Detail

Bio

Hans J Haubold is an astrophysicist pursuing research on solar neutrinos and special functions of mathematical physics, in cooperation with AM Mathai.

Constantino Tsallis is a Greek-born Brazilian physicist working in Rio de Janeiro at CBPF, Brazil. He introduced the notions known as Tsallis entropy and Tsallis statistics in his 1988 paper 'Possible generalization of Boltzmann–Gibbs statistics', published in the *Journal of Statistical Physics.*

Funding

The researchers acknowledge partial financial support by Brazilian agencies CNPq and Faperj.

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Personal Response

What are the main barriers to the widespread acceptance of Tsallis entropy?

" Most of the resistance that existed 30 years ago has disappeared, surely as a consequence of the ever increasing validating evidence (experimental, observational, numerical, analytical). The original resistance is no surprise: it has always lurked in science when a longstanding paradigm is shaken. A crucial point is the fact that it takes long for the ensemble of physicists to embody that the indices q by no means are mere fitting parameters but can instead be deduced from first-principle physical foundations unless intractable mathematical difficulties emerge. Analytical illustrations of this fact gradually accumulate in diverse complex systems, and will ineluctably end up becoming self-evident for most scientists disposed to face scientific challenges. "

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